

Symbolic Polynomial Equation Solver

Solver for linear polynomial equations with symbolic coefficients compatible with Symbolic Math Toolbox and Polynomial Toolbox.

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March 3, 2002

sxab

Purpose Equation solver for polynomial matrices with symbolic coefficients.

Syntax $X = \text{sxab}(A, B, p)$
 $X = \text{sxab}(A, B, p, \text{degree})$
 $[X, R] = \text{sxab}(A, B, p)$

Description The command
 $X0 = \text{sxab}(A, B, p)$
 finds a particular solution of the linear matrix polynomial equation $X(p)A(p) = B(p)$ using Sylvester matrix method. Here A and B are polynomial matrices in symbolic variable p .
 The command
 $X0 = \text{sxab}(A, B, p, \text{DEGREE})$
 seeks a solution $X0$ of degree DEGREE . If DEGREE is not specified then a solution of minimum overall degree is computed.
 The command
 $[X0, K] = \text{sxab}(A, B, p)$
 also computes the left null-space of A so that all the solutions to $XA = B$ may be parametrized as $X = X_0 + TK$ where T is an arbitrary polynomial matrix.

Examples **Example 1 Equation with A(2×2) and B(1×2)**

Variable of A and B and symbolic coefficients must be declared as symbolic objects:

```
syms p a0 a1 a2 a3
A=[p+a0 a1*p^2;a2 a3*p]
B=[p^2+(a0+a2)*p+a2 a1*p^3+a3*p^2+a3*p]

A =
[ p+a0, a1*p^2]
[ a2, a3*p]
B =
[ p^2+(a0+a2)*p+a2, a1*p^3+a3*p^2+a3*p]

X=sxab(A,B,p)

X =
[ p, p+1]

simplify(X*A-B)

ans =
[ 0, 0]
```

Example 2 Equation with a nonsquare A

```
A=[p+a0 a1*p^2;a2 a3*p;3*p p^2+1]
A =
[ p+a0, a1*p^2]
[ a2, a3*p]
[ 3*p, p^2+1]

B=[p*(p+a0)+a1*p*a2+15*p
a1*p^3+a1*p^2*a3+5*p^2+5;p+a0+(p+a2)*a2+3*p^2
a1*p^2+(p+a2)*a3*p+p*(p^2+1)]
B =
[ p*(p+a0)+a1*p*a2+15*p,
a1*p^3+a1*p^2*a3+5*p^2+5]
[ p+a0+(p+a2)*a2+3*p^2,
a1*p^2+(p+a2)*a3*p+p*(p^2+1)]

X=sxab(A,B,p)
X =
[ p, a1*p, 5]
[ 1, p+a2, p]
```

Example 3 Solution of a given degree

```
syms p a
A=[a*p ;1 ;p+1]
B=(a+1)*p^2+p+a
A =
[ p*a]
[ 1]
[ p+1]
B =
(a+1)*p^2+p+a

X=sxab(A,B,p,2)
X =
[ 0, (-a+1)*p+(a+1)*p^2, a]
```

Example 4 General solution

```
syms s a b
A=[a*s (a+b)*s;s+1 5*s;a*b*s 1]
B=[a*b*s^3+s^2+(a+1)*s 6*s^2+(a+b)*s]
A =
[ a*s, (a+b)*s]
[ s+1, 5*s]
[ a*b*s, 1]
B =
[ s^3*a*b+s^2+(a+1)*s, (a+b)*s+6*s^2]

[X,K]=sxab(A,B,s)
X =
[ 1, s, s^2]
K =
[ s+1-5*a*b*s^2, -a*s+a*b*(a+b)*s^2, (-a-
b)*s+(4*a-b)*s^2]

simplify((X+K)*A-B)
ans =
[ 0, 0]
```

Example 5 Visualizing the solution process

Solution is in fact found in a series of steps by searching within a certain range of degrees. The whole process may be displayed by setting the *verbose level* equal to 'yes'.

```
A =
[ -5*s^2+s+1, -s^2+5*s-4]
[ -3*s^2+2, -5*s^2+2*s-3]
[ -4*s^2+3*s+4, -11*s^2-6*s-1]
B =
[ s^2+2*s-3, -6*s^3-2*s^2-5*s-5]
[ -6*s^3+7*s-5, -s^3-7*s^2-6*s]

X=sxab(A,B,s)
SXAB: Minimum expected degree: 1./Maximum expected degree:
6.
SXAB: Seek minimum degree solution.
SXAB: Attempt # 1 Degree 1. No.
SXAB: Attempt # 2 Degree 2. No.
SXAB: Attempt # 3 Degree 3. Yes.
SXAB: Solution of degree 3 was found.
```

Algorithm

The equation is solved through Sylvester matrix method which means that a Sylvester matrix corresponding to polynomial matrix $A = A_0 + A_1 \cdot p + A_2 \cdot p^2 + \dots + A_n \cdot p^n$ is constructed concerning degree of the right side of the equation. Then the right matrix division is used to find solution.

Diagnostics

The macro displays an error messages if

- there are not enough input arguments
- input matrices are not of class *sym* or *double*

dimensions of input matrices are inconsistent

See also

saxbyc, sxaybc symbolic solution of other types of Diophantine equation
xab, axbyc, xaybc numerical solvers

saxbyc,sxaybc

Purpose	Diophantine equation solver for polynomial matrices with symbolic coefficients.
Syntax	$[X, Y] = \text{saxbyc}(A, B, C, p)$ $[X, Y, R, S] = \text{saxbyc}(A, B, C, p)$ $[X, Y] = \text{sxaybc}(A, B, C, p)$ $[X, Y, R, S] = \text{sxaybc}(A, B, C, p)$
Description	<p>The command</p> $[X, Y] = \text{saxbyc}(A, B, C, p)$ <p>solves the linear polynomial matrix equation $A(p)X(p) + B(p)Y(p) = C(p)$ where A, B and C are polynomial matrices in variable p. X and Y are unknown matrices. If no solution exists then constant matrices X and Y filled with NaNs are returned. It solves the equation through the Sylvester matrix method.</p> <p>The command</p> $[X, Y, R, S] = \text{saxbyc}(A, B, C, p)$ <p>additionally computes the right null-space of $[A \ B]$ so that all the solutions to $AX + BY = C$ may be parametrized as</p> $X = X_0 + RT$ $Y = Y_0 + ST$ <p>where T is an arbitrary polynomial matrix.</p> <p>The command</p> $[X, Y] = \text{sxaybc}(A, B, C, p)$ <p>solves the linear polynomial matrix equation $X(p)A(p) + Y(p)B(p) = C(p)$ where A, B and C are polynomial matrices in variable p while X and Y are unknown matrices. If no solution exists then constant matrices X and Y filled with NaNs are returned. It solves the equation through the Sylvester matrix method.</p> <p>The command</p> $[X, Y, R, S] = \text{sxaybc}(A, B, C, p)$ <p>additionally computes the left null-space of $[A; B]$ so that all the solutions to $XA + YB = C$ may be parametrized as</p> $X = X_0 + TR$ $Y = Y_0 + TS$ <p>where T is an arbitrary polynomial matrix.</p>
Examples	<p>Example 1 Scalar Diophantine equation</p> <p>First symbolic objects for variable and symbolic coefficients must be declared:</p>

```

syms s a b
A=[s^2+a*s+b]
A =
s^2+a*s+b
B=[5*s^2 s^2+1]
B =
[ 5*s^2, s^2+1]
C=1
C =
1

```

To solve Diophantine equation type:

```

[X, Y]=saxbyc(A, B, C, s)
X =
0
Y =
[ -1/5]
[ 1]

```

Example 2 General solution of matrix equation

Consider three polynomial matrices:

```

A=[a*s+1 s+a*b]
A =
[ a*s+1, s+a*b]

B=[a*s+5*b s^2]
B =
[ a*s+5*b, s^2]

C=1
C =
1

```

To find general solution of Diophantine equation type:

```

[X0, Y0, R, S]=saxbyc(A, B, C, s)
X0 =
[ -1/(-1+5*b) ]
[ 0 ]
Y0 =
[ 1/(-1+5*b) ]
[ 0 ]
R =
[ -b*(-5+a^2)/(-1+a^2*b) ]
[ -a*(-1+5*b)/(-1+a^2*b) ]
S =
[ 1 ]
[ 0 ]

```

Choosing arbitrary parameter T equal to 1 we obtain:

```

X=X0+R
X =
[ -1/(-1+5*b)-b*(-5+a^2)/(-1+a^2*b) ]
[ -a*(-1+5*b)/(-1+a^2*b) ]

Y=Y0+S
Y =
[ 1/(-1+5*b)+1 ]
[ 0 ]

```

And to check the solution:

```
simplify(A*X+B*Y-C)  
ans =  
0
```

Algorithm After suitable preparation the macros call routine sxab.

Diagnostics The macro displays an error messages if

- there is wrong number of input arguments
- there is wrong number of output arguments
- dimensions of input matrices are inconsistent

See also sxab symbolic solution of polynomial equation

matcoef**Purpose** Coefficient matrix of a symbolic polynomial matrix.**Syntax** `Acoef = matcoef (A , p)`**Description** The command
`Acoef = matcoef (A , p)`
returns matrix of block coefficients of the polynomial matrix A in symbolic variable p .**Examples** Example Polynomial matrix $A(2 \times 2)$ to matrix of coefficients

Introduce symbolic objects

`syms x a b`and form a polynomial matrix in x :`A=[a*x (a+b)*x^2;a*b^2 x]``A =``[a*x, (a+b)*x^2]`
`[a*b^2, x]`

To find its matrix of coefficients, just type:

`Acoef=matcoef(A,x)``Acoef =``[0, 0, a, 0, 0, a+b]`
`[a*b^2, 0, 0, 1, 0, 0]`matrix coefficients at x^0 The coefficients naturally contain the other two symbolic variables: a, b .If A is considered a polynomial matrix in a , the coefficient contain x and b :`Acoef=matcoef(A,a)``Acoef =``[0, b*x^2, x, x^2]`
`[0, x, b^2, 0]`**Algorithm** Uses macro `matdeg` to find matrix degree and searches for coefficients by differentiating the input matrix in appropriate number of steps.**Diagnostics** The macro displays an error messages if
• there is wrong number of input arguments
• input matrix is not of class `sym` or `double`**See also** `poly2sym` (only for numerical coefficients)

matdeg**Purpose** **Various degree matrices of a symbolic polynomial matrix.****Syntax** $D = \text{matdeg}(A, p)$
 $D = \text{matdeg}(A, p, 'r')$
 $D = \text{matdeg}(A, p, 'c')$
 $D = \text{matdeg}(A, p, 'all')$ **Description** The command
 $D = \text{matdeg}(A, p)$
returns the degree of polynomial matrix in symbolic variable p .
The command
 $D = \text{matdeg}(A, p, 'r')$
returns a column vector containing the row degrees of A .
The command
 $D = \text{matdeg}(A, p, 'c')$
returns a row vector containing the column degrees of A .
The command
 $D = \text{matdeg}(A, p, 'all')$
returns a matrix containing the degrees of the entries of A .**Examples** **Example Degrees of polynomial matrix A(2×2)**

First symbolic objects must be declared:

```
syms x a b
```

Consider polynomial matrix in variable s :

```
A=[a*x (a+b)*x^2;a*b^2 x]
```

```
A =
```

```
[      a*x, (a+b)*x^2]
[      a*b^2,      x]
```

To find degree of A just type:

```
matdeg(A, x)
```

```
ans =
      2
```

Row degrees of A :

```
matdeg(A, x, 'r')
```

```
ans =
      2
      1
```

Column degrees of A :

```
matdeg(A, x, 'c')
```

```
ans =
      1      2
```

Degrees of the entries of A :

```
matdeg(A, x, 'all')
```

```
ans =
      1      2
      0      1
```

Algorithm	Uses maple function <i>degree</i> to find degrees of polynomials.
Diagnostics	The macro displays an error messages if <ul style="list-style-type: none">• there is wrong number of input arguments• input matrix is not of class <i>sym</i> or <i>double</i>• the option is not valid (different 'c', 'r' or 'all')
See also	pol/deg degrees of a standard polynomial matrix

smsylv

Purpose Create the Sylvester matrix corresponding to a symbolic polynomial matrix.

Syntax $S = \text{smsylv}(A, p, k)$

Description For a symbolic polynomial matrix $A = A_0 + A_1p + A_2p^2 + \dots + A_n p^n$, the command $S = \text{smsylv}(A, p, k)$

creates Sylvester matrix

$$S = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 & \cdots & A_n & 0 & 0 & 0 & 0 \\ 0 & A_0 & A_1 & A_2 & A_3 & \cdots & A_n & 0 & 0 & 0 \\ 0 & 0 & & \cdots & \ddots & & \cdots & 0 & & \\ 0 & 0 & 0 & 0 & A_0 & A_1 & A_2 & A_3 & & A_n \end{bmatrix}$$

block row 1
block row 2
...
block row k+1

(k is the number of zero blocks in each block row)

Examples Example Sylvester matrix for A(3x2)

Symbolic objects:

```
syms s a b
```

Consider polynomial matrix in variable s :

```
A=[a*s^2+b (a-b)*s;a*b*s (a*b)^2*s^2;a*s-b-5 s^2+a*b]
```

A =

```
[ a*s^2+b, (a-b)*s]
[ a*b*s, a^2*b^2*s^2]
[ a*s-b-5, s^2+a*b]
```

Sylvester matrix with 0 zero blocks:

```
S=smsylv(A,s,0)
```

S =

```
[ b, 0, 0, a-b, a, 0]
[ 0, 0, a*b, 0, 0, a^2*b^2]
[ -5-b, a*b, a, 0, 0, 1]
```

With one zero block:

```
S=smsylv(A,s,1)
```

S =

```
[ b, 0, 0, a-b, a, 0, 0, 0]
[ 0, 0, a*b, 0, 0, a^2*b^2, 0, 0]
[ -5-b, a*b, a, 0, 0, 1, 0, 0]
[ 0, 0, b, 0, 0, a-b, a, 0]
[ 0, 0, 0, 0, a*b, 0, 0, a^2*b^2]
[ 0, 0, -5-b, a*b, a, 0, 0, 1]
```

Algorithm Uses macro *matcoef* to find matrix of symbolic coefficients and then creates Sylvester matrix after scheme mentioned above.

Diagnostics The macro displays an error messages if

- there is wrong number of input arguments
- input matrix is not of class *sym*

See also `pol/sylv` Sylvester matrix of a standard polynomial matrix